Physics Letters A ••• (••••) •••–•••

Contents lists available at ScienceDirect

Physics Letters A

www.elsevier.com/locate/pla

Stationary wave packets in the Kerr nonlinear media with imaginary harmonic potential and linear gain

F.Kh. Abdullaev [a,b], R.M. Galimzyanov [b,*]

[a] Physical-Technical Institute of the Academy of Sciences, Bodomzor Yoli street 2-b, 100084, Tashkent-84, Uzbekistan
[b] CCNH, Universidade Federal do ABC, 09210-170, Santo André, Brazil

ARTICLE INFO

Article history:
Received 31 January 2018
Received in revised form 19 April 2018
Accepted 23 April 2018
Available online xxxx
Communicated by V.A. Markel

Keywords:
1D BEC
Gross–Pitaevskii equation
Imaginary potential
Variational analysis
Super-Gaussian ansatz
Nonlinear Kerr media

ABSTRACT

The existence of stationary wave packets in the nonlinear Kerr media with an imaginary harmonic potential and a linear gain is investigated. By employing a variational approach the existence of stable bright solitons is shown for the case of a defocusing nonlinearity. In focusing nonlinear media, the bright solitons have been shown to be unstable. The predictions of variational approach are confirmed by numerical simulations of the full modified NLS equation. The predicted stationary localized wave packets can be observed in a quasi-one-dimensional BEC with an imaginary optical potential and atoms feeding.

1. Introduction

The problem of nonlinear wave processes in media with non-Hermitian (mainly a parity-time (PT) symmetric) parameters like potentials, nonlinearities etc. attracted a great attention in recent time [1,2].

Existence and stability of solitons in PT-symmetric systems have been investigated in works [3–6]. Solitons in non-PT symmetric complex potentials have been recently studied in [7–11]. Dynamics of solitons in real potentials with inhomogeneous gain has been considered in [12–14].

Very interesting result has been obtained in [15], where evolution of a wave packet under action of an imaginary quadratic potential (non PT-symmetric) was considered. Authors found an exact solution of the Schrödinger equation with such potential. The width of this packet asymptotically with time tends to the stationary value. However such a packet is decaying and its amplitude is going to zero in time. The experimental work with atomic beams in such potential demonstrates the existence of a "nonspreading" wave packet [16].

In the case of BEC affect of atomic interactions is important. It is interesting to investigate effect of interaction between atoms and gain on the existence of stationary "nonspreading" wave packets.

Here for this purpose we consider a theoretical model based on the NLS equation with cubic nonlinearity, gain and imaginary harmonic potential. This problem has been of a general interest since it described nonlinear waves in optical and cold atomic systems. We use the Gross–Pitaevskii equation in description of the wave function of a quasi one dimensional BEC with imaginary trap and gain of the atoms number. In the case of the nonlinear optics, the media with the Kerr nonlinearity and spatially inhomogeneous distribution of the linear part of the imaginary refraction index and linear amplification can be considered as such a system.

Important finding (see below) is that in such imaginary potential it is possible the existence of stable bright matter wave solitons for the case of a repulsive interaction (defocusing mean field nonlinearity), while in the case of attractive interactions (focusing nonlinearity) the bright solitons are unstable. Some analogues were found earlier, namely the existence of nonlinear localized modes in a nonlinear medium with non-Hermitian harmonic potential [17], a quasi-stationary bright solitons in the 2D Kerr nonlinear media with quintic dissipation [18]. Also, it should be noted that the existence of a bright soliton solution in the slab waveguide with defocusing nonlinearity, absorbing boundary and localized nonlinear gain in the core has been recently shown in work [19]. Here the authors employed the possibility to manipulate the sign of the effective nonlinearity in such systems, using properties of the spectrum of a linear non-Hermitian system.

The paper is organized as follows. In section 2 we give a mathematical model and get governing Gross–Pitaevskii (GP) equation to

* Corresponding author.
E-mail address: ravil@uzsci.net (R.M. Galimzyanov).

https://doi.org/10.1016/j.physleta.2018.04.051
0375-9601/© 2018 Published by Elsevier B.V.

describe the dynamics of the system. Then we formulate a variational approach (VA) corresponding to our governing equation and describing the behavior of parameters of the wave packet in time. In section 4 and further we present results of numerical simulations and make some conclusions.

2. The model

As the model, we will consider matter waves in a quasi-one-dimensional BEC with the atoms feeding, loaded in the imaginary quadratic potential. Such configuration can be realized by the condensation of two-level atoms in the laser field and strong transverse confinement trap potential. The interaction between resonant laser field and the ground state of the atom can be described by a complex optical potential [20–22]

$$V_{opt} = \frac{1}{\hbar}\frac{d_e^2 E^2(X)}{\Delta + i\gamma_0/2}, \tag{1}$$

where $E(X)$ is the electric field, d_e is the dipole moment of the atom, Δ is the detuning of the laser beam frequency from the transition frequency between levels, γ_0 is the loss rate from the excited state to the noninteracting case. This potential becomes imaginary for $\Delta = 0$.

Corresponding Gross–Pitaevskii equation describing this system is:

$$i\hbar\psi_T + \frac{\hbar^2}{2M}\psi_{XX} + iM\alpha^2\frac{\omega_0^2}{2}X^2\psi - g_{1D}|\psi|^2\psi - i\rho\psi = 0, \tag{2}$$

where ρ is the atoms feeding rate parameter, ω_0 is a parameter of the optical potential, $g_{1D} = 2\hbar\omega_\perp a_S$, $\omega_\perp$ is the transverse trap frequency, a_S is the S-wave atomic scattering length, $a_S > 0$ ($a_S < 0$) corresponds to BEC with repulsive (attractive) interaction between atoms respectively. Introducing the dimensionless variables

$$t = \frac{T\omega_0}{2},\ x = \frac{X}{l},\ l = \sqrt{\frac{\hbar}{M\omega_0}},\ \delta = \frac{2\rho}{\hbar\omega_0},\ u = \sqrt{2|a_S|}\psi,$$
$$\gamma = a_S/|a_S|,$$

we get the equation in the form of a modified nonlinear Schrödinger equation:

$$iu_t + u_{xx} - \gamma|u|^2u = (-i\alpha^2x^2 + i\delta)u, \tag{3}$$

where factors α and δ are the imaginary harmonic potential strength and gain correspondingly. This equation also describes the propagation of electromagnetic waves in a nonlinear 1D Kerr medium with gain and a harmonic modulation of the imaginary part of a linear refraction index [23].

The linear case without gain ($\gamma = \delta = 0$) was considered in work [15]. The exact solution which was named as "nonspreading wave packet" has the form

$$u(x,t) = \sqrt{\frac{1/\pi}{\cosh(\widetilde{\beta}t)}}\exp\left(-\frac{1}{2}\widetilde{\alpha}x^2\tanh(\widetilde{\beta}t)\right), \tag{4}$$

where $\widetilde{\alpha} = \alpha\exp(-i\pi/4)$, $\widetilde{\beta} = 2\alpha\exp(i\pi/4)$. The time dependent width is $\delta x(t) = [Re(\widetilde{\alpha}\tanh(\widetilde{\beta}t))]^{-1/2}$ and it achieves the stationary value for $2\alpha t \gg 1$: $\delta x_0 = 2^{\frac{1}{4}}/\alpha$.

3. Variational approach

To find localized stable states we employ the Lagrangian formalism to describe the behavior of unknown solution parameters. Let us introduce the following super-Gaussian trial function as an ansatz

$$u(x,t) = A(t)\exp\left(-\frac{1}{2}\left(\frac{x}{a(t)}\right)^{2m(t)} + ib(t)x^2 + i\varphi(t)\right), \tag{5}$$

where A, a, b, m, φ are the amplitude, width, chirp, super-Gaussian index and linear phase, respectively.

For nonconservative systems being described by Eq. (3) the Lagrangian formalism with Euler–Lagrange equations can be formulated as follows

$$\frac{\partial\mathcal{L}}{\partial u^*} - \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial u_t^*}\right) - \frac{d}{dx}\left(\frac{\partial\mathcal{L}}{\partial u_x^*}\right) = \frac{\partial\mathcal{L}_R}{\partial u^*}. \tag{6}$$

Here $\mathcal{L}$ is the Lagrangian density of a conservative part of the system given by

$$\mathcal{L} = \frac{i}{2}(u_tu^* - u_t^*u) - |u_x|^2 - \frac{1}{2}\gamma|u|^4 \tag{7}$$

and

$$\frac{\partial\mathcal{L}_R}{\partial u^*} = R = (-i\alpha^2x^2 + i\delta)u \tag{8}$$

is the right side of Eq. (3), namely a dissipation force.

To apply a variational approach we proceed from the ansatz (5). Let unknown function $u(x,t)$ be expressed in some parameters $\eta_i(t)$ (unknown too). To obtain equations for the unknown parameters we should consider spatially averaged Lagranjian

$$L(\eta_i(t),t) = \int_{-\infty}^{\infty}\mathcal{L}\left(\hat{u}(x,t,\eta_i), \hat{u}^*(x,t,\eta_i)\right)dx.$$

Then a system of Euler–Lagrange equations for parameters η_i reads [24]

$$\frac{\partial L}{\partial\eta_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\eta}_i}\right) = \int_{-\infty}^{\infty}dx\left(R\frac{\partial u^*}{\partial\eta_i} + R^*\frac{\partial u}{\partial\eta_i}\right). \tag{9}$$

In the case of our parametrization (5) of unknown solution, conservative part of the averaged Lagranjian takes the following form

$$L = -A^2a\left(\frac{a^2}{m}\Gamma\left(\frac{3}{2m}\right)b_t + \frac{1}{m}\Gamma\left(\frac{1}{2m}\right)\phi_t + \frac{m}{a^2}\Gamma\left(2 - \frac{1}{2m}\right) + \frac{a^2}{m}\Gamma\left(\frac{3}{2m}\right)4b^2 + \frac{1}{m}2^{-\frac{1}{2m}}\Gamma\left(\frac{1}{2m}\right)\frac{\gamma}{2}A^2\right). \tag{10}$$

Substituting this expression into the Euler–Lagrange equations (9) we get ordinary differential equations for unknown parameters A, a, b, m and ϕ:

$$\frac{d}{dt}\left(\Gamma\left(\frac{1}{2m}\right)\frac{A^2a}{m}\right) = \left(-2\frac{\Gamma\left(\frac{3}{2m}\right)}{\Gamma\left(\frac{1}{2m}\right)}\alpha^2a^2 + 2\delta\right)\left(\Gamma\left(\frac{1}{2m}\right)\frac{A^2a}{m}\right), \tag{11}$$

$$\frac{d}{dt}\left(\Gamma\left(\frac{3}{2m}\right)\frac{A^2a^3}{m}\right) = \left(8b - 2\frac{\Gamma\left(\frac{5}{2m}\right)}{\Gamma\left(\frac{3}{2m}\right)}\alpha^2a^2 + 2\delta\right)\left(\Gamma\left(\frac{3}{2m}\right)\frac{A^2a^3}{m}\right), \tag{12}$$

$$\frac{a^2}{m}\Gamma\left(\frac{3}{2m}\right)\frac{db}{dt} = \frac{m}{a^2}\Gamma\left(2 - \frac{1}{2m}\right) - \frac{4}{m}\Gamma\left(\frac{3}{2m}\right)a^2b^2 + \frac{2^{-\frac{1}{2m}}}{m}\Gamma\left(\frac{1}{2m}\right)\frac{\gamma}{4}A^2, \tag{13}$$

$$\left(\frac{2}{m}-\frac{3}{2m^2}\Psi\Big(\frac{3}{2m}\Big)+\frac{1}{m^2}\Psi\Big(\frac{1}{2m}\Big)+\frac{1}{2m^2}\Psi\Big(2-\frac{1}{2m}\Big)\right)\frac{m}{a^2}\Gamma\Big(2-\frac{1}{2m}\Big)-\left(-\frac{2}{m}+\frac{3}{2m^2}\Psi\Big(\frac{3}{2m}\Big)-\frac{3}{2m^2}\Psi\Big(\frac{1}{2m}\Big)-\frac{\ln(2)}{m^2}\right)\frac{\gamma A^2}{4}\frac{2^{-\frac{1}{2m}}}{m}\Gamma\Big(\frac{1}{2m}\Big)=0, \tag{14}$$

$$\frac{1}{m}\Gamma\Big(\frac{1}{2m}\Big)\frac{d\phi}{dt}=-\frac{2m}{a^2}\Gamma\Big(2-\frac{1}{2m}\Big)-\frac{5}{4}\frac{2^{-\frac{1}{2m}}}{m}\Gamma\Big(\frac{1}{2m}\Big)\gamma A^2. \tag{15}$$

As seen from equations (11)–(15), parameter ϕ does not enter into the first four equations for A, a, b, m. And so we can limit ourselves by considering only equilibrium states of these four parameters:

$$\frac{dA}{dt}=0,\quad \frac{da}{dt}=0,\quad \frac{db}{dt}=0,\quad \frac{dm}{dt}=0.$$

Equations for the fixed points of the ODE take the following form:

$$-2\frac{\Gamma\Big(\frac{3}{2m}\Big)}{\Gamma\Big(\frac{1}{2m}\Big)}\alpha^2a^2+2\delta=0, \tag{16}$$

$$8b-2\frac{\Gamma\Big(\frac{5}{2m}\Big)}{\Gamma\Big(\frac{3}{2m}\Big)}\alpha^2a^2+2\delta=0, \tag{17}$$

$$\frac{m}{a^2}\Gamma\Big(2-\frac{1}{2m}\Big)-\frac{4}{m}\Gamma\Big(\frac{3}{2m}\Big)a^2b^2+\frac{2^{-\frac{1}{2m}}}{m}\Gamma\Big(\frac{1}{2m}\Big)\frac{\gamma}{4}A^2=0, \tag{18}$$

$$\left(\frac{2}{m}-\frac{3}{2m^2}\Psi\Big(\frac{3}{2m}\Big)+\frac{1}{m^2}\Psi\Big(\frac{1}{2m}\Big)+\frac{1}{2m^2}\Psi\Big(2-\frac{1}{2m}\Big)\right)\frac{m}{a^2}\Gamma\Big(2-\frac{1}{2m}\Big)-\left(-\frac{2}{m}+\frac{3}{2m^2}\Psi\Big(\frac{3}{2m}\Big)-\frac{3}{2m^2}\Psi\Big(\frac{1}{2m}\Big)-\frac{\ln(2)}{m^2}\right)\frac{\gamma A^2}{4}\frac{2^{-\frac{1}{2m}}}{m}\Gamma\Big(\frac{1}{2m}\Big)=0. \tag{19}$$

Solving these equations we get two sets of stationary points. The first is:

$$b_{s1}=\left(\frac{\Gamma(5/2m)\Gamma(1/2m)}{\Gamma(3/2m)^2}-1\right)\frac{\delta}{4},$$
$$a_{s1}=\frac{1}{\alpha}\sqrt{\frac{\Gamma(1/2m)}{\Gamma(3/2m)}\delta},$$
$$A_{s1}=\sqrt{\frac{4m2^{\frac{1}{2m}}}{\gamma\Gamma(\frac{1}{2m})}\Big(\frac{4}{m}\Gamma\Big(\frac{3}{2m}\Big)a_{s1}^2b_{s1}^2-\frac{m}{a_{s1}^2}\Gamma\Big(2-\frac{1}{2m}\Big)\Big)}.$$

Stationary value of the super-Gaussian index m_{s1} is calculated numerically by solving Eqs. (16)–(19). The rest parameters b, a, A are obtained from the above formulas.

The second set corresponds to the solution, stationary amplitude of which tends to zero ($A \to 0$). So corresponding stationary values of the parameters are obtained from the same equations (16)–(19), supposing amplitude A to be zero:

$$8b_{s2}-2\Big(\frac{\Gamma(\frac{5}{2m})}{\Gamma(\frac{3}{2m})}-\frac{\Gamma(\frac{3}{2m})}{\Gamma(\frac{1}{2m})}\Big)\alpha^2a_{s2}^2=0,$$
$$\frac{m}{a_{s2}^2}\Gamma\Big(2-\frac{1}{2m}\Big)-\frac{4}{m}\Gamma\Big(\frac{3}{2m}\Big)a_{s2}^2b_{s2}^2=0,$$
$$\frac{2}{m}-\frac{3}{2m^2}\Psi\Big(\frac{3}{2m}\Big)+\frac{1}{m^2}\Psi\Big(\frac{1}{2m}\Big)+\frac{1}{2m^2}\Psi\Big(2-\frac{1}{2m}\Big)=0.$$

The above equations give the following solution for stationary values for the second set:

$$A_{s2}=0,$$
$$m_{s2}=1,$$
$$a_{s2}=\frac{(2m)^{1/4}}{\sqrt{\alpha}}\left(\frac{\Gamma(5/2m)}{\Gamma(3/2m)}-\frac{\Gamma(3/2m)}{\Gamma(1/2m)}\right)^{-1/4}\times\left(\frac{\Gamma(2-1/2m)}{\Gamma(3/2m)}\right)^{1/8},$$
$$b_{s2}=\frac{\alpha}{4}(2m)^{1/2}\left(\frac{\Gamma(5/2m)}{\Gamma(3/2m)}-\frac{\Gamma(3/2m)}{\Gamma(1/2m)}\right)^{1/2}\times\left(\frac{\Gamma(2-1/2m)}{\Gamma(3/2m)}\right)^{1/4}. \tag{20}$$

In Fig. 1 behaviors of stationary values of variational parameters a, b, A are shown when varying super-Gaussian parameter m at different values of α in the case of defocusing media ($\gamma > 0$). Corresponding stationary values of these parameters obtained numerically by solving the governing PDE (3) (horizontal lines) are also depicted there. Intersections of the curves with corresponding asymptotes, obtained numerically from PDE, give an appropriate value of m in a variational description of a given parameter. One can see that for values of dissipation α in the range 0.15–0.25 an optimal value of m is about $m = 1$ for parameters a, b and around $m = 1.3$ for A. This is right for values of the dissipation α in the interval (0.15–0.25). If we select the optimal value $m = 1.3$ for A, variational stationary values of the width and the chirp will deviate from the results of the numerical simulations of PDE by ≈ 0.15 relative units. So guided by above and the simplicity of resulting equations, hereinafter we will consider the case when the super-Gaussian index is supposed to be 1 ($m = 1$), i.e. we use a Gaussian ansatz. In studying variational approach for another range of dissipation α one may be need to consider another value of super-Gaussian index m. In the case of $m = 1$ the system of equations for the parameters A, a, b takes the following form:

$$\frac{dA}{dt}=(\delta-2b)A,$$
$$\frac{da}{dt}=4ab-\alpha^2a^3,$$
$$\frac{db}{dt}=\frac{1}{a^4}-4b^2+\frac{\gamma A^2}{2\sqrt{2}a^2}. \tag{21}$$

Then stationary values of the variational parameters are determined as:

$$b_{s1}=\frac{\delta}{2},\quad a_{s1}=\frac{\sqrt{2\delta}}{\alpha},\quad A_{s1}=\sqrt{\frac{\sqrt{2}}{\gamma}\Big(4\frac{\delta^3}{\alpha^2}-\frac{\alpha^2}{\delta}\Big)} \tag{22}$$

and

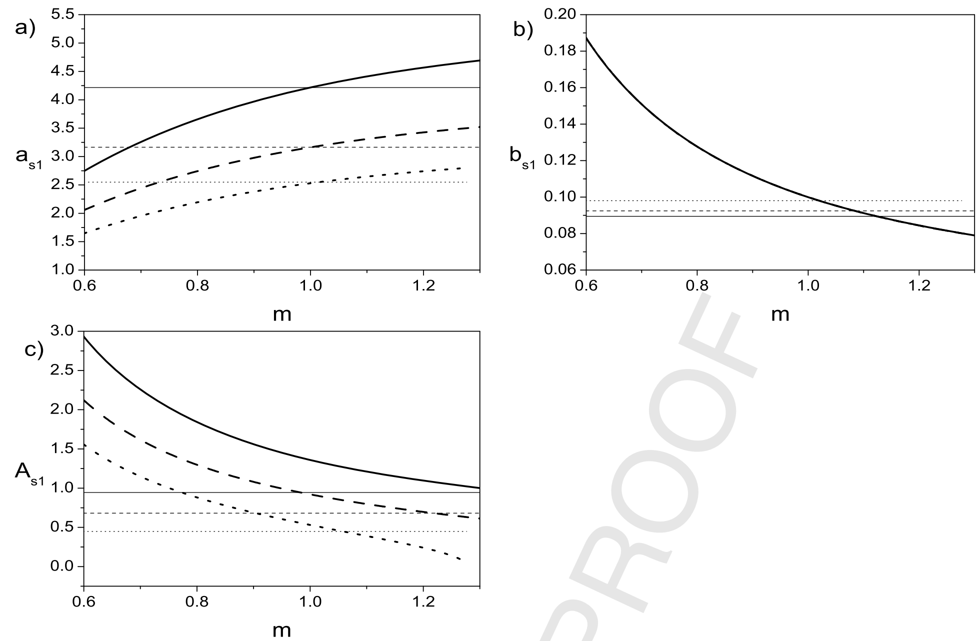

Fig. 1. Dependencies of the stationary variational parameters a_{s1}, b_{s1} and A_{s1} on super-Gaussian index m for the case of a defocusing medium ($\gamma = 1$) for different dissipation α: a) the dependencies of width a_{s1}, b) the dependencies of chirp b_{s1}, c) the dependencies of amplitude A_{s1}. Horizontal lines are for stationary values (asymptotical in time) obtained from numerically solved governing PDE (3). In the three plots solid lines are for $\alpha = 0.15$, dash lines for $\alpha = 0.2$ and dot lines for $\alpha = 0.25$. Everywhere gain $\delta = 0.2$.

$$b_{s2} = \frac{\alpha}{2\sqrt{2}}, \quad a_{s2} = \frac{2^{1/4}}{\sqrt{\alpha}}, \quad A_{s2} = 0. \tag{23}$$

Note that the value of the gain $\delta = \alpha/\sqrt{2}$ corresponds to the bifurcation point for the case of defocusing media ($\gamma > 0$). In the case of focusing media ($\gamma < 0$) the bifurcation point is shifted a little and $\delta \lessapprox \alpha/\sqrt{2}$.

The linear stability analysis of the VA system for stationary values (22).

Let us look for the solutions of the system (21) in the form

$$A = A_s + A_1, \; b = b_s + b_1, \; a = a_s + a_1,$$

where A_1/A_s, b_1/b_s, $a_1/a_s \ll 1$. The system of equations for small corrections A_1, b_1, a_1 is:

$$\begin{aligned} \frac{da_1}{dt} &= -4\delta a_1 + \frac{4\sqrt{2\delta}}{\alpha} b_1, \\ \frac{db_1}{dt} &= -\Lambda a_1 - 4\delta b_1 + \frac{\gamma \alpha^2 A_s}{2\sqrt{2}\delta} A_1, \\ \frac{dA_1}{dt} &= -2A_s b_1, \end{aligned} \tag{24}$$

where

$$\Lambda = \frac{\alpha^5}{\sqrt{2}\delta^{5/2}} \left(1 + \frac{\gamma A_s^2 \delta}{2\sqrt{2}\alpha^2}\right).$$

Looking for the solutions A_1, a_1, $b_1 \sim \exp(\lambda t)$, we obtain the following characteristic equation:

$$\begin{aligned} &\lambda^3 + 8\delta\lambda^2 + 16\delta^2 \left(1 + \frac{\Lambda}{2\delta\sqrt{2\delta}\alpha} + \frac{\gamma\alpha^2 A_s^2}{16\sqrt{2}\delta^3}\right)\lambda \\ &\quad + 2\sqrt{2}\gamma\alpha^2 A_s^2 = 0. \end{aligned} \tag{25}$$

Real positive eigenvalues correspond to the unstable solutions of the VA system. Below we will study the different cases of linear, defocusing and focusing nonlinear media.

Linear media, $\gamma = 0$. When $\gamma = 0$ the nonlinear term in Eq. (3) vanishes and we get a linear medium with a harmonic imaginary (dissipative) potential of strength α and homogeneous gain δ. In this case there only a single stationary point exists, determined by α (dissipation) to δ (gain) *ratio* equal to:

$$\alpha/\delta = \sqrt{2}, \tag{26}$$

as follows from the solution of variational equations Eq. (21) when $\gamma = 0$. Stationary value of amplitude A is determined by all the parameters of an initial wave packet. In other case, when the above equality is not observed, only divergence of the solution amplitude A or its vanishing is observed. At the same time parameters width a and chirp b tend to the stationary values Eq. (23)

$$b_s = b_{s2}, \quad a_s = a_{s2} \tag{27}$$

irrespective of the amplitude dynamics.

It should be noted the fact that a stationary value of the wave packet width a_{s2} obtained from our variational analysis of the governing equation Eq. (3) using ansatz Eq. (5) coincides with the one determined by Berry's solution Eq. (4).

Defocusing Kerr media, $\gamma > 0$. When $\gamma > 0$ all factors in characteristic equation Eq. (25) are of the positive sign, so eigenvalues of the VA problem are to be negative. In this case stable stationary points Eq. (22) exist provided that $\alpha/\delta \leq \sqrt{2}$. Stationary values of the parameters Eq. (22) are:

$$b_s = b_{s1}, \quad a_s = a_{s1}, \quad A_s = A_{s1}. \tag{28}$$

If not and $\alpha/\delta > \sqrt{2}$ the amplitude of solution A is going to zero ($A_s = 0$), but the parameters chirp and width keep other stationary nonzero values Eq. (23)

$$b_s = b_{s2}, \quad a_s = a_{s2}. \tag{29}$$

So the point Eq. (26) is a bifurcation point separating solutions Eq. (22) and Eq. (23).

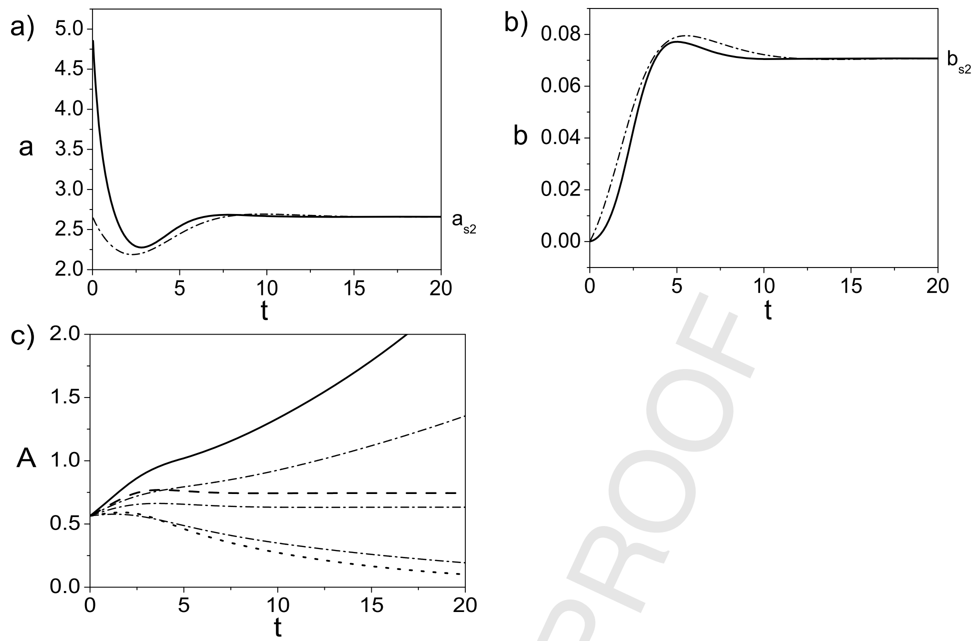

Fig. 2. Time dependencies of the solution parameters a, b and A calculated from ODE Eq. (21) and PDE Eq. (3) for different values of gain $\delta = 0.2,\ \alpha/\sqrt{2}$ and 0.05 in the case of linear medium ($\gamma = 0$) are shown. Evolutions of the parameters calculated from the ODE are presented by dash-dot lines. a) Evolutions of the width a, b) evolutions of the chirp b. The parameters a and b calculated by solving PDE are presented by solid lines. All the three dependencies calculated by ODE for different δ coincide. The same is for ones calculated by PDE. c) Time dependencies of the amplitude A calculated by PDE are presented for $\delta = 0.2$ (solid line), $\delta = \alpha/\sqrt{2}$ (dash line) and $\delta = 0.05$ (dot line). All the dependencies of parameter A calculated by ODE are presented by dash-dot lines for all values of the gain δ. Everywhere dissipation $\alpha = 0.2$.

Focusing Kerr media, $\gamma < 0$. In this case, for any dissipation α to gain δ ratio there exists a divergent set of the solution parameters:

$$b_s \to 0, \quad a_s \to 0, \quad A \to \infty. \tag{30}$$

Provided that $\alpha/\delta \geq \sqrt{2}$ there exists also a nonstable set of the solution parameters Eq. (28) with finite amplitude A. Evolution of the solution in this case depends on parameters of the initial wave packet including its amplitude A_0. If A_0 is greater than some value, the amplitude A diverges and the case Eq. (30) is realized. When A_0 is less, the solution parameters evolve to the case Eq. (23):

$$b_s = b_{s2}, \quad a_s = a_{s2}, \quad A_s \to 0.$$

4. Numerical results

Our numerical simulations of evolution of a single pulse in the linear and nonlinear Kerr media are based on the dimensionless governing equation Eq. (3) and a set of ordinary equations Eq. (21) (to describe variational parameters). In all PDE calculations of the governing equation, we have employed a wave packet Eq. (4) from [15,16] as an initial wave packet (with $t_0 = 0.1$). This expression corresponds to the initial state in the form of the chirped Gaussian wave packet and is the particular choice of the initial variational ansatz with $A_0 = 0.564,\ b_0 = 0.0001,\ a_0 \approx 5$. In numerical simulations of the governing PDE Eq. (3), amplitude A of the localized wave packet $u(x,t)$, its squared width a and chirp b at time t are calculated as

$$A = max|u(x,t)|,$$
$$a^2 = \frac{\int_{-\infty}^{\infty} x^2 |u(x,t)|^2 dx}{\int_{-\infty}^{\infty} |u(x,t)|^2 dx},$$
$$b = \frac{Im\left(\int_{-\infty}^{\infty} u(x,t)^2 u(x,t)_x^{*2} dx\right)}{\int_{-\infty}^{\infty} |u(x,t)|^4 dx}. \tag{31}$$

First we considered dynamics of the pulse in a linear medium ($\gamma = 0$) with different values of the gain δ.

When $\gamma = 0$ variational parameters width a and chirp b tend in time to stationary values Eq. (23) irrespective of the amplitude dynamics. The behavior of A in VA is determined by right side of the first equation of set (21). So if $\delta < 2b_{s2}$ ($\delta > 2b_{s2}$) the amplitude A is going to zero (to infinity). When the gain is equal to $\delta = \alpha/\sqrt{2}$ stabilization of the amplitude is observed. It should be noted that this stationary amplitude is not equal to A_{s1} because of that Eq. (22) determining A_{s1} is not applicable when $\gamma = 0$. In this case the value of stationary A is determined by all the initial values of parameters A, a, b governing the wave packet dynamics.

In Fig. 2 evolutions of the pulse parameters chirp b, width a and amplitude A in the linear case, $\gamma = 0$ are shown. The evolutions of variational parameters a, b being described by set (21) for different values of gain $\delta = 0.2,\ \alpha/\sqrt{2}$ and 0.05 are depicted by dash-dot lines. Ones obtained from solution of the governing PDE (3) making use of Eq. (31) are presented by solid lines. Everywhere dissipation $\alpha = 0.2$. One can see that curves describing ODE evolutions of a, b for different gain δ coincide. The cause of it is that equations (21) determining dynamics of the VA parameters a and b do not contain parameter δ and are independent on the dynamics of A. Coincidence of the PDE calculated time dependencies of parameters a, b for different gain δ as in the case of ODE time dependencies for a, b shows that when m equals to 1 VA description of a, b is in a good agreement with PDE simulations.

Evolutions of the amplitude depend on the value of δ, or rather the gain to dissipation ratio, δ/α. In Fig. 2c time dependencies of variational amplitude A (ODE calculated) and the dependencies obtained from PDE using Eq. (31) differ noticeably. But it should be noted the fact that stabilization of states obtained by simulation of PDE Eq. (3) and stabilization of ones obtained from variational ODE calculations Eq. (21) are realized at the same value of the ratio $\alpha/\delta = \sqrt{2}$.

Then we considered the case of a defocusing medium ($\gamma > 0$).

In Fig. 3 behaviors of the pulse parameters chirp b, width a and amplitude A are shown for a defocusing medium, $\gamma = 1$. As seen,

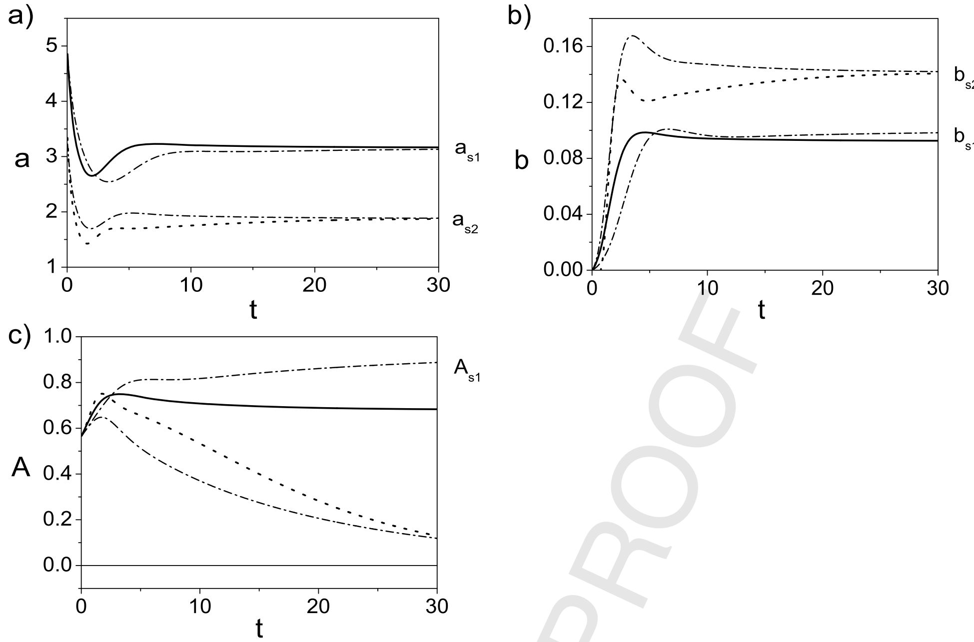

Fig. 3. Time dependencies of the pulse parameters chirp b, width a and amplitude A in the case of a defocusing medium ($\gamma = 1$) for two values of the dissipation to gain ratio: $\alpha/\delta < \sqrt{2}$ (stable solutions) and $\alpha/\delta > \sqrt{2}$ (decaying solutions). a) Evolutions of width a; b) evolutions of chirp b; c) evolutions of amplitude A. All the dependencies of parameters $a,\ b,\ A$ calculated by ODE are presented by dash-dot lines. The dependencies of ones calculated by PDE for $\alpha/\delta < \sqrt{2}$, ($\alpha = 0.2,\ \delta = 0.2$) (stable evolution) are presented by solid lines and by dot-lines for $\alpha/\delta > \sqrt{2}$, ($\alpha = 0.4,\ \delta = 0.2$) (decaying evolution).

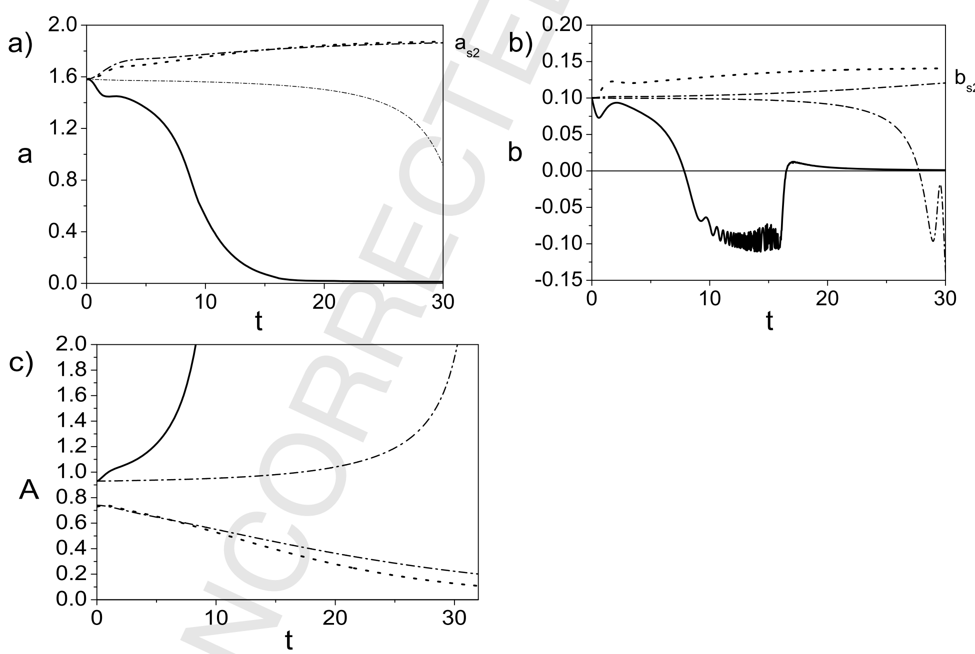

Fig. 4. Time dependencies of the pulse parameters chirp b, width a and amplitude A are shown for the case of a focusing medium ($\gamma = -1$) for two values of the initial wave packet amplitude, $A_0 > A_{crit}$ and $A_0 < A_{crit}$ for $\alpha = 0.4$, $\delta = 0.2$. a) Evolutions of width a; b) evolutions of chirp b; c) evolutions of amplitude A. All the dependencies of parameters $a,\ b,\ A$ calculated by ODE are presented by dash-dot lines. PDE calculated evolutions of the divergent dependencies ($A_0 > A_{crit}$) are presented by solid lines and evolutions of the decaying dependencies ($A_0 < A_{crit}$) by dot lines.

when the existence condition $\alpha/\delta \le \sqrt{2}$ is observed, all pulse parameters $a,\ b,\ A$ possess stable stationary values. Ones calculated by ODE are equal to $a_{s1},\ b_{s1},\ A_{s1}$ correspondingly. If the existence condition is not observed, the amplitudes A (ODE and PDE calculated) tend to zero. At the same time ODE calculated pulse parameters a and b tend to other stable stationary values, viz. a_{s2} and, b_{s2} correspondingly. Everywhere the dependencies calculated by ODE are presented by dash-dot lines.

At last we considered the case of a focusing medium ($\gamma < 0$).

In Fig. 4 evolutions of the wave packet parameters chirp b, width a and amplitude A obtained from PDE simulation of Eq. (3) and ODE calculation Eq. (21) are shown for the case of a focusing medium, $\gamma = -1$. All ODE calculated dependencies are depicted by dash-dot lines. When $\gamma = -1$ for any value of dissipation α and gain δ there exists a divergent solution of amplitude $A \to \infty$ with the values of width a and chirp b tending to zero. When $\alpha/\delta > \sqrt{2}$ a solution emerges with decaying amplitude A and stable stationary values of width a and chirp b. One can see that when value of the initial wavepacket amplitude A_0 is greater than

Please cite this article in press as: F.Kh. Abdullaev, R.M. Galimzyanov, Stationary wave packets in the Kerr nonlinear media with imaginary harmonic potential and linear gain, Phys. Lett. A (2018), https://doi.org/10.1016/j.physleta.2018.04.051

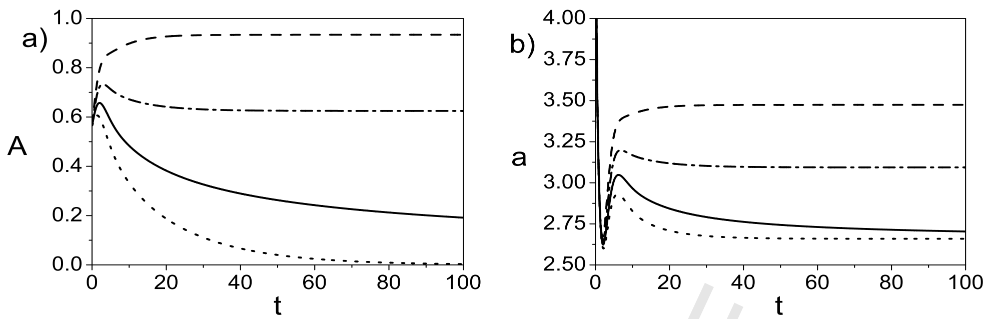

Fig. 5. Time dependencies of the PDE calculated pulse parameters amplitude A and width a are shown for the case of a defocusing medium ($\gamma = 1$) at different set of dissipation α and gain δ being close to the bifurcation point $\delta = \alpha/\sqrt{2}$. a) Evolutions of amplitude A; b) evolutions of width a. Dash lines are for $\delta = \alpha/\sqrt{2} + 0.1$, dash dot lines are for $\delta = \alpha/\sqrt{2} + 0.05$, solid lines are for $\delta = \alpha/\sqrt{2}$, and dot lines are for $\delta = \alpha/\sqrt{2} - 0.05$. Everywhere $\alpha = 0.2$. Only the parameter δ is varied at different sets.

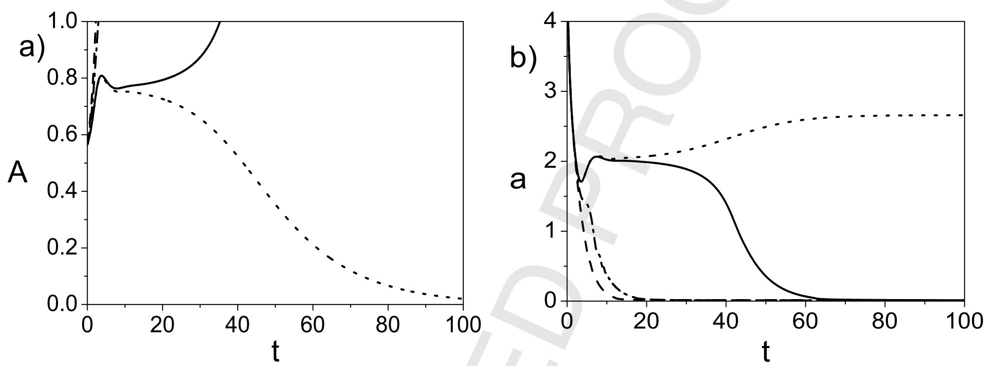

Fig. 6. Time dependencies of the pulse parameters amplitude A and width a in the case of a focusing medium ($\gamma = -1$) for different set of dissipation α and gain δ being close to the bifurcation point $\delta = \alpha/\sqrt{2}$. a) Evolutions of amplitude A; b) evolutions of width a. Dash lines are for $\delta = \alpha/\sqrt{2} + 0.05$, dash dot lines are for $\delta = \alpha/\sqrt{2}$, solid lines are for $\delta = \alpha/\sqrt{2} - 0.059$, dot lines are for $\delta = \alpha/\sqrt{2} - 0.06$. Everywhere $\alpha = 0.2$ and only parameter δ is varied in different sets.

some critical value A_{crit} being determined by characteristic equation Eq. (25) and the rest parameters a and b, the solution diverges. For given $\alpha = 0.4$, $\delta = 0.2$, $a_0 = 1.58, b_0 = 0.1$ critical amplitude $A_{crit} \simeq 0.925$. When initial amplitude $A_0 < A_{crit}$, the solution amplitude tends to zero but parameters width a and chirp b tend to stationary values a_{s2} and b_{s2}.

As seen VA description of the wave-packet evolutions are in agreement with the simulations of PDE, Eq. (3).

In Fig. 5 time dependencies of the PDE calculated pulse parameters b, a, A in a defocusing medium ($\gamma = 1$) are shown for different set of dissipation α and gain δ being close to the bifurcation point $\delta = \alpha/\sqrt{2}$. As seen the value of stationary amplitude A is equal to zero ($A_{s1} = 0$) at the bifurcation point $\alpha/\delta = \sqrt{2}$. The greater the gain δ to dissipation α ratio δ/α the greater stationary value of amplitude A.

In Fig. 6 time dependencies of the PDE calculated pulse parameters b, a, A are shown for the case of a focusing medium ($\gamma = -1$) for different values of dissipation α and gain δ when their ratio δ/α is close to $\sqrt{2}$. The bifurcation point for this case is shifted and determined by $\delta = \alpha/\sqrt{2} - 0.059$. One can see that evolutions near this point are very unstable.

In Figs. 7, 8 and 9 evolutions of the wave packet profiles are shown to demonstrate the processes of stabilization ($\alpha = 0.2$, $\delta = 0.2$) and decay ($\alpha = 0.4$, $\delta = 0.2$) in a defocusing medium ($\gamma = 1$) and divergence of wave packets ($\alpha = 0.4$, $\delta = 0.2$) in a focusing medium ($\gamma = -1$). As was mentioned for a focusing medium, when the initial wavepacket amplitude $A_0 > A_{crit}$ being determined by parameters a, b, the solution diverges. For given $\alpha = 0.4$, $\delta = 0.2$, $a_0 = 1.58, b_0 = 0.1$ critical amplitude $A_{crit} \simeq 0.925$.

One can see that profiles of the wave packet keep their forms during the evolution. The radiation is not observed for all considered cases. The wave packet profiles have been calculated by simulation of the governing equation Eq. (3).

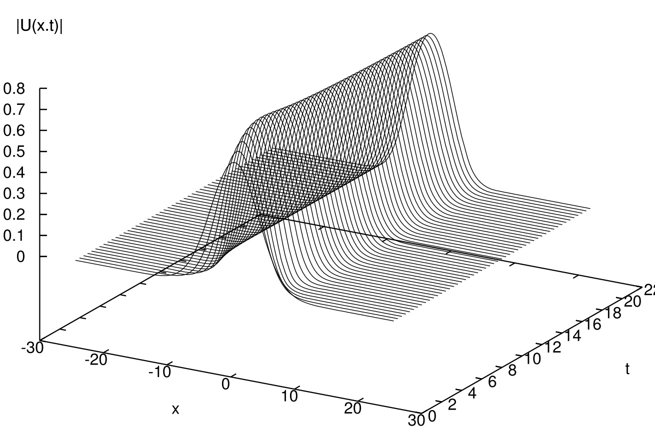

Fig. 7. Stabilization of the wave packet profile is shown for the case of a defocusing medium $\gamma = 1$ when $\alpha/\delta < \sqrt{2}$, ($\alpha = 0.2$, $\delta = 0.2$).

5. Conclusion

In conclusion we have investigated the dynamics of the wave packets in media with cubic nonlinearity and an imaginary quadratic potential with addition of gain. We have carried out variational analysis of evolution of the wave packet and numerical simulations of the governing equation Eq. (3). We have found out the existence of stable stationary wave packets in linear and defocusing media. The latter case corresponds to the existence of bright solitons in media with the defocusing nonlinearity. For the case when the nonlinearity corresponds to a focusing medium, fixed stationary points for the wave packet parameters obtained from VA, have been revealed to be unstable. In the case of a linear medium, we have shown that asymptotical behaviors of parame-

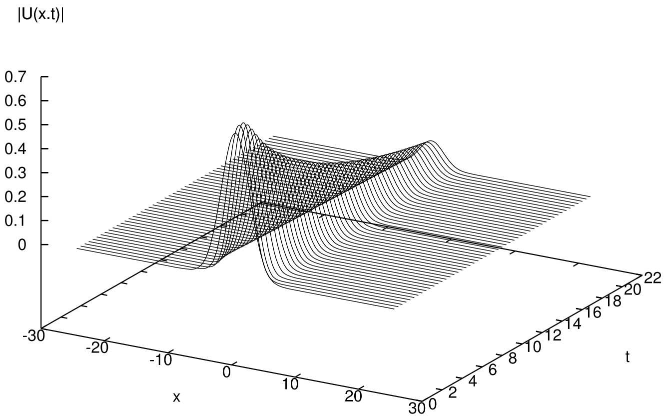

Fig. 8. Decay of the wave packet profile is shown for the case of a defocusing medium ($\gamma = 1$) when $\alpha/\delta > \sqrt{2}$, ($\alpha = 0.4,\ \ \delta = 0.2$).

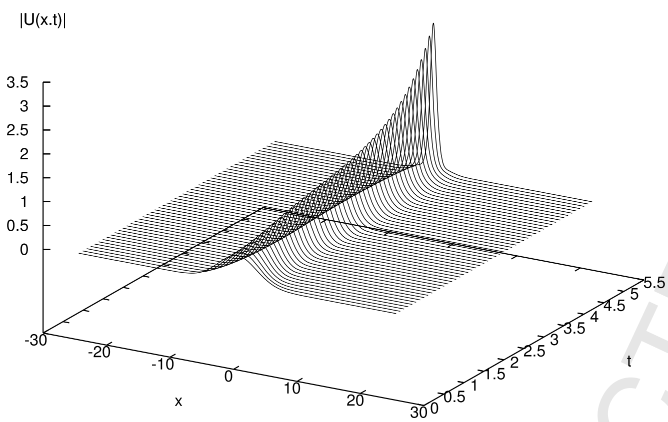

Fig. 9. Divergence of the wave packet profile is shown for the case of a focusing medium $\gamma = -1$ when $\alpha/\delta > \sqrt{2}$ and $A_0 > A_{crit}$. The parameters $\alpha = 0.4, /\delta = 0.2,\ \ a_0 = 1.58, b_0 = 0.1,\ \ A_{crit} \simeq 0.925$.

ters width a and chirp b of the wave packet are independent on evolution of the amplitude A (the wave power or the number of atoms for the BEC). The predicted stationary states can be observed in the experiment with atomic BEC in a cigar-shaped trap with the imaginary quadratic potential, atoms feeding and the repulsive interactions between atoms.

Acknowledgements

Authors are grateful to E.N. Tsoy and B.B. Baizakov for useful comments and suggestions. F.A. is grateful to CNPq (Brasil) grant for financial support in the framework of the PVE programme. R.G. acknowledges support by Fund for Fundamental Researches of the Uzbek Academy of Sciences (Award No FA-F2-004).

References

[1] V.V. Konotop, J. Yang, D.A. Zezyulin, Rev. Mod. Phys. 88 (2016) 035002.
[2] S.V. Suchkov, A.A. Sukhorukov, J. Huang, S.V. Dmitriev, C. Lee, Yu.S. Kivshar, Laser Photonics Rev. 10 (2016) 177.
[3] Z.H. Musslimani, K.G. Makris, R. El-Ganainy, D.N. Christodoulides, Phys. Rev. Lett. 100 (2008) 030402.
[4] F.Kh. Abdullaev, V.V. Konotop, M. Salerno, A.V. Yulin, Phys. Rev. E 82 (2010) 056606.
[5] F.Kh. Abdullaev, Y.V. Kartashov, V.V. Konotop, D.A. Zezyulin, Phys. Rev. A 83 (2011) 041805R.
[6] S. Nixon, L. Ge, Y. Yang, Phys. Rev. A 85 (2002) 023822.
[7] E.N. Tsoy, I.M. Allayarov, F.Kh. Abdullaev, Opt. Lett. 39 (2014) 4215.
[8] V.V. Konotop, D.A. Zezyulin, Opt. Lett. 39 (2014) 5535.
[9] Y. Kominis, Phys. Rev. A 92 (2015) 063849.
[10] Y. Kominis, Opt. Commun. 334 (2015) 265.
[11] S.D. Nixon, J. Yang, Stud. Appl. Math. 136 (2016) 459.
[12] H. Sakaguchi, B.A. Malomed, Phys. Rev. E 77 (2008) 056606.
[13] Y. Kominis, S. Douglas, P. Papagiannis, K. Hizanidis, Phys. Rev. A 85 (2012) 063801.
[14] Y. Kominis, P. Papagiannis, S. Douglas, Opt. Express 20 (2012) 18165.
[15] M.V. Berry, D.H.J. O'Dell, J. Phys. A, Math. Gen. 31 (1998) 2093.
[16] R. Stutzle, et al., Phys. Rev. Lett. 95 (2005) 110405.
[17] D.A. Zezyulin, G.L. Alfimov, V.V. Konotop, Phys. Rev. A 81 (2010) 013606.
[18] A. Alexandrescu, V.M. Perez-Garcia, Phys. Rev. A 73 (2006) 053610.
[19] B. Midya, V.V. Konotop, Phys. Rev. Lett. 119 (2017) 033905.
[20] D.O. Chudesnikov, V.P. Yakovlev, Laser Phys. 1 (1991) 110.
[21] M. Oberthaler, et al., Phys. Rev. Lett. 77 (1996) 4980.
[22] M.A. Efremov, et al., Laser Phys. 13 (2003) 995;
M.V. Fedorov, et al., J. Exp. Theor. Phys. 97 (2003) 522.
[23] G.P. Agrawal, Yu.S. Kivshar, Optical Solitons, AP, 2004.
[24] B.A. Malomed, Prog. Opt. 43 (2002) 69.

Please cite this article in press as: F.Kh. Abdullaev, R.M. Galimzyanov, Stationary wave packets in the Kerr nonlinear media with imaginary harmonic potential and linear gain, Phys. Lett. A (2018), https://doi.org/10.1016/j.physleta.2018.04.051